Math 116 **ACTIVITY 2:** Working with the (definite) integral. The fundamental theorem and some additional properties

Why

THe definite integral is an important piece of the calculus, used for expressing total change, total effect of some ongoing change. It is important to anchor your understanding of its meaning and the methods of calculation—particularly the use of antiderivatives through the fundamental theorem.

LEARNING OBJECTIVES

1. Be able to set up a definite integral for a "total change" situation and an "average value" situation
2. Understand the use of the Fundamental Theorem for calculation of an integral
3. Understand and be able to use the additional properties for rewriting and calculating integrals

CITERIA

1. Success in working as a team and in fulfilling the team roles.
2. Success in involving all members of the team in the conversation.
3. Success in completing the exercises

RESOURCES

1. The course syllabus
2. The team role desk markers (handed out in class for use during the semester)
3. 40 minutes

PLAN

1. Select roles, if you have not already done so, and decide how you will carry out steps 2 and 3 (5 minutes)
2. Work through the group exercises given here - be sure everyone understands all results & procedures(25 minutes)
3. Assess the team's work and roles performances and prepare the Reflector's and Recorder's reports including team grade (5 minutes).

DISCUSSION

We have been working with the integral $\int_a^b f(t)\ dt$ as "the total change produced by a rate $f(t)$ over an interval $t = a$ to $t = b$"; it also can be considered the *signed* area beneath (" above" counts as negative) the graph of $y = f(t)$ from $t = a$ to $t = b$. The formal definition is in terms of the limit of Riemann sums, and we also looked at approximation of the integral using Riemann sums without taking the limit.

Properties of the integral that we have we developed/studied/defined:

$\int_a^a f(x)\ dx = 0$

$\int_a^b f(x)\ dx = -\int_b^a f(x)\ dx$

$\frac{d}{dx}\int_a^x f(t)\ dt = f(x)$

The fundamental theorem (the big deal): $\int_a^b f(x)\ dx = F(b) - F(a)$ for any antiderivative $F(x)$ of $f(x)$

The *average value* of a function $f(x)$ from $x = a$ to $x = b$ is given by $\frac{1}{b-a}\int_a^b f(x)\ dx$

The fundamental theorem brings back our interest in antiderivatives:
Here are the anitderivative rules we have so far:
A. The basic antiderivative rules

1. Constant $\int k\ dx = kx + C$ for any constant k

2. Power (nice case): $\int x^n \, dx = \dfrac{x^{n+1}}{n+1} + C$ when $n \neq -1$
3. Power (less-nice case): $\int x^{-1} \, dx = \int \dfrac{1}{x} \, dx = \ln|x| + C$
4. Exponential: $\int e^{ax} \, dx = \dfrac{1}{a} e^{ax} + C$ for any constant a
5. Sine: $\int \sin ax \, dx = -\dfrac{1}{a} \cos ax + C$ for any constant a
6. Cosine: $\int \cos ax \, dx = \dfrac{1}{a} \sin ax + C$ for any constant a

B: Reduction rules

1. Constant coefficient: $\int kf(x)dx = k \int f(x)dx$ for any constant k
2. Sum/difference: $\int f(x) \pm g(x)dx = \int f(x)dx \pm \int g(x)dx$

Now we add four more properties of the integral: The first two come from properties of antiderivatives, as a result of the fundamental theorem. The third is based on the fact that the integral is, basically, a sum, so it can be broken up into pieces. The fourth goes back to the idea of area as base times height - the height *between* two graphs can be expressed as "*top* − *bottom*".

1. Constant coefficient: $\int_a^b kf(x) \, dx = k \int_a^b f(x) \, dx$ for any constant k
2. Sum/difference of functions: $\int_a^b f(x) \pm g(x) \, dx = \int_a^b f(x) \, dx \pm \int_a^b g(x) \, dx$
3. Splitting up interval: $\int_a^b f(x) \, dx = \int_a^c f(x) \, dx + \int_c^b f(x) \, dx$ for any c between a and b (This is useful when there is a different formula for the function on different parts of the interval from a to b

4. The area between the graphs of $y = f(x)$ and $y = g(x)$ from a to b, if $f(x) \geq g(x)$, is given by $\int_a^b f(x) - g(x) \, dx$ (We integrate "top" − "bottom". If the curves cross (so formulas for "bottom" and "top" change), we have to split up the area, using #3.). The "area between the graphs" (no limits given) is the area between points where the graphs cross and we need to find the x values to use for our limits of integration.

An example for #3: If $f(x) = \begin{cases} 3x-4 & \text{if } x \leq 4, \\ x+4 & \text{otherwise} \end{cases}$ then we have a problem calculating $\int_2^6 f(x) \, dx$ because we don't have a nice formula for $f(x)$, and certainly don't have a nice formula for an antiderivative. However, we *do* have a formula for $f(x)$ when $x \leq 4$ and we do have a formula for $f(x)$ when $x > 4$ so we split up the integral and use the appropriate formula for each piece:

$$\int_2^5 f(x) \, dx = \int_2^4 f(x) \, dx + \int_4^5 f(x) \, dx = \int_2^4 3x - 4 \, dx + \int_4^5 x + 4 \, dx = \frac{3}{2}x^2 - 4x\Big|_2^4 + \frac{1}{2}x^2 + 4x\Big|_4^5$$
$$= \left(\frac{3}{2}(4)^2 - 4(4)\right) - \left(\frac{3}{2}(2)^2 - 4(2)\right) + \left(\frac{1}{2}(5)^2 + 4(5)\right) - \left(\frac{1}{2}(4)^2 + 4(4)\right) = (8) - (-2) + (32.5) - (24) = 18.5$$

An example for #4: The area between the graphs of $f(x) = 2x$ and $g(x) = x^2$. The graphs cross when $f(x) = g(x)$. that is, when $2x = x^2$ (at $x = 0$ and at $x = 2$). Since $2x > x^2$ for x between 0 and 2, the top curve is the graph of f, the bottom curve is the graph of g, so the area is $\int_0^2 2x - x^2 \, dx = x^2 - \frac{1}{3}x^3\Big|_0^2 =$ $\left(2^2 - \frac{1}{3}(2)^3\right) - \left(0^2 - \frac{1}{3}(0)^3\right) = \frac{16}{3}$

EXERCISE

1. Calculate:

(a) $\int_1^5 6x^2 - \frac{3}{x^2} + \frac{1}{x}\ dx$

(b) The change in size (total mass) of a population from time $t = 2$ weeks to time $t = 4$ weeks if the growth rate at time t weeks is given by $f(t) = .025e^{.5t}$ g/week. (round to 3 decimal places)

(c) The average value of $f(x) = \sin\left(\frac{\pi}{4}x\right)$ from $x = 0$ to $x = 4$

2. Calculate: $\int_1^2 2x^3 - x\ dx$ and $\int_2^4 2x^3 - x\ dx$ and use these directly to give $\int_1^4 2x^3 - x\ dx$

3. (a) Find the area between the graphs of $f(x) = x$ and $f(x) = x^2$ from $x = 0$ to $x = 1$

   (b) Find the area between the graphs of $f(x) = x$ and $f(x) = x^2$ from $x = 1$ to $x = 3$

   (c) What is the total area (not *signed* area but actual area) between the graphs from $x = 0$ to $x = 3$?

READING ASSIGNMENT (in preparation for next class)
In your text, read section 5.5: Integration techniques: Substitution (actually this is “antiderivative” calcualtions)

SKILL EXERCISES:(hand in - individually - with assignment for this week)
p. 369 #71, 74, 83, p.378 #5, 8, 15, 16, 22, 23